V = volume containing the region of interest in a problem
v = velocity vector
v_i = velocity component in the direction of the coordinate x_i
x_i = coordinates
Y_l^* = exact solution for one of the dependent variables $\mathcal{P}$, v_i, T, c_A
Y_l = approximate solution for one of the dependent variables

Greek Letters

α = angle of inclination of the plane walls
ϵ_0 = the initial error, the amount by which an approximate solution fails to satisfy the initial conditions
ϵ_V = the interior error, the amount by which an approximate solution fails to satisfy the differential equations
ϵ_S = the boundary error, the amount by which an approximate solution fails to satisfy the boundary conditions on S
μ = viscosity
ρ = density
ϕ_j = the jth member of a set of position dependent trial functions
Φ_j = the jth member of a set of general trial functions

Subscripts

A, B = chemical species in a binary system
i, j, k = summation indices
n = upper limit on summation indices
P = pressure function
S = on the surface
V = within the volume
v = velocity
0 = initial state

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Manuscript received July 19, 1963; revision received January 8, 1964; paper accepted January 10, 1964. Paper presented at A.I.Ch.E. Buffalo meeting.

Turbulent Film Condensation

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Nusselt's laminar film theory of condensation (*10*) is simple and capable of predicting the heat transfer coefficient in many cases. However the actual heat transfer coefficients are found to be somewhat higher for fluids with moderate and high Prandtl numbers, while the heat transfer coefficients observed for the liquid metals (small Prandtl number) are considerably lower than Nusselt's prediction. These lead one to investigate the effect of turbulent transports in condensation because the thickness of condensate film can become large in reality.

In the past few attempts have been made in this direction, notably by Seban (*12*), Roshenow et al. (*11*), and Dukler (*3*). Seban adopted the well-known universal logarithmic velocity distributions for a velocity profile in the condensate film. With an additional assumption of eddy thermal diffusivity being equal to eddy kinematic viscosity the heat transfer analysis was carried out by an approximate analytical method. The theory was extended by Rohsenow et al. to include the positive interfacial shear stress. In view of a recent work by Lee (*6*) the use of logarithmic velocity distribution seems to be valid only in the case of a large positive interfacial shear stress. Recently Dukler presented a more ambitious approach of actually solving the velocity profile using the eddy viscosity expressions of Deissler and von Karman. However his velocity profile suffers from the physical inconsistencies, such as the discontinuity at the intersection of two regions and the failure to satisfy the interfacial shear stress correctly. Furthermore his heat transfer results for small Prandtl number range do not seem physically plausible owing to neglecting the molecular thermal conductivity with respect to eddy conductivity.

The purpose of this paper is to recompute the heat transfer coefficients for turbulent Nusselt's model without introducing a priori assumption on the magnitude of transport coefficients. The method of solution is that developed by the author (*6*), in which a smooth velocity profile with correct interfacial shear stress is obtained. In contrast to Dukler's work the correct heat transfer coefficients are still much larger than the liquid metals data of Misra and Bonilla (*9*). This is not unexpected because the Nusselt's model does not account for the following considerations: nonvanishing interfacial shear stress, inertia effects, convective heat transfer and subcooling, interfacial waves and

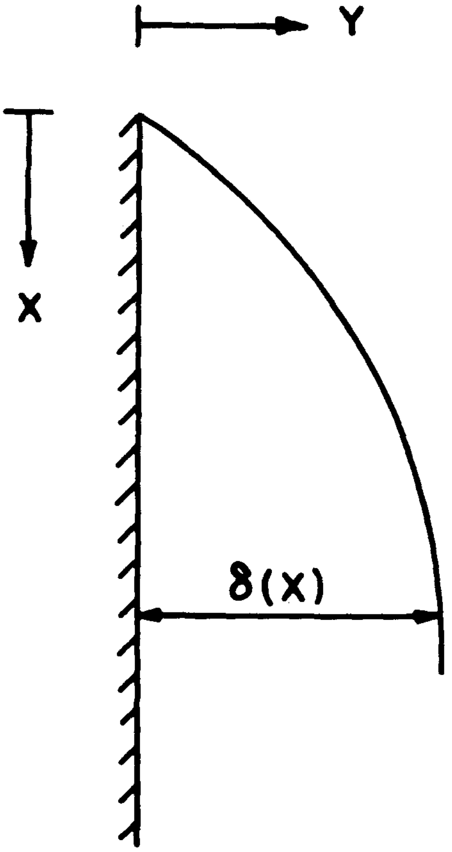

Fig. 1. Model and coordinates.

rippling, interfacial thermal resistance, and contact thermal resistance.

FORMULATION OF PROBLEM

Consider a physical model of film condensation on a vertical semi-infinite plate, where the x axis is parallel and the y axis is perpendicular to the plate, as shown in Figure 1. This paper is restricted to the physical model that was originally adopted by Nusselt. Consider only the balance of acceleration due to gravity and retardation due to drag on the plate. Neglect the inertia effects and the interfacial shear stress. Consider the conduction type of heat energy transport to be dominant, and neglect the convective heat transfer and subcooling. However what is different from the Nusselt's formulation is the inclusion of turbulent transports in the form of eddy diffusivity.

Under the usual assumption of the constancy of physical properties within the temperature range involved the equations of motion and heat energy in the condensate film can be written as

$$(\nu + \nu_e)\frac{\partial u}{\partial y} = g(\delta - y) \tag{1}$$

$$(\kappa + \kappa_e)\frac{\partial T}{\partial y} = \kappa\left(\frac{\partial T}{\partial y}\right)_o \tag{2}$$

where ν_e and κ_e denote the eddy kinematic viscosity and eddy thermal diffusivity. The boundary conditions imposed are

$$\left.\begin{array}{l} u = 0 \\ T = T_w \end{array}\right\} \text{at } y = 0, \quad x \geqq 0$$
$$T = T_s \quad \text{at } y = \delta, \quad x \geqq 0$$

To define the a priori undermined δ one must evoke the physical phenomenon of condensation which is expressed as

$$\frac{\lambda}{C_p}\frac{\partial}{\partial x}\int_0^\delta u\,dy - \kappa\left(\frac{\partial T}{\partial y}\right)_o = 0 \tag{3}$$

Equations (1), (2), and (3) constitute the system of governing equations which are reducible to the Nusselt's laminar problem, if ν_e and κ_e were dropped. For the eddy kinematic viscosities, Deissler's expression (*2*)

$$\nu_e = n^2 uy\,[1 - \exp(-n^2uy/\nu)]. \quad 0 \leqq y \leqq y^* \tag{4}$$

and von Karman's expression (*13*)

$$\nu_e = K^2\,|(\partial u/\partial y)^3/(\partial^2 u/\partial y^2)^2|, \quad y^* \leqq y \tag{5}$$

are used, where n, K, and y^* are empirical constants. Deissler suggested a value of $y^* = 26\,\nu/\sqrt{\tau_w/\rho}$ for pipe flows; however a slightly different value of y^* will be used here (*5, 6*).

SOME MANIPULATIONS

Accept the additional assumption that the eddy thermal diffusivity is a constant multiple of eddy kinematic viscosity; that is $\nu_e/\kappa_e = \alpha =$ constant. Since ν_e vanishes at the wall, $y^* = C\,\nu/\sqrt{\delta g}$ from (1) (C is a constant). Along the x axis the precise value of y^* cannot be assigned ab initio owing to its dependency on δ. Nevertheless only two cases exist depending on the value of δ; that is $\delta \leqq y^*$ and $\delta \geqq y^*$.

Case I: $\delta \leqq y^*$

The differential equations for velocity and temperature profiles can be obtained from (1), (2), and (4)

$$\frac{\partial u_1}{\partial y} = \frac{(g/\nu)(\delta - y)}{1 + (n^2/\nu)\,u_1\,y\,[1 - \exp(-n^2u_1y/\nu)]}, \quad \delta \leqq y^* \tag{6}$$

$$\frac{\partial T}{\partial y} = \frac{(\partial T/\partial y)_o}{1 + N_{Pr}(n^2/\nu)u_1y\,[1 - \exp(-n^2u_1y/\nu)]\,\alpha}, \quad \delta \leqq y^* \tag{7}$$

where u in Deissler's region is denoted by u_1. Equation (7) can be integrated at once to give

$$\frac{T - T_w}{\Delta T} = \frac{1}{I_1}\int_0^y \frac{dy}{1 + N_{Pr}\,(n^2/\nu)u_1y[1 - \exp(-n^2u_1y/\nu)]\alpha} \tag{8}$$

where I_1 denotes the integral of (8) from 0 to δ.

To specify δ Equation (3) will be written in an alternate form

$$\left[\int_0^\delta \left(\frac{\partial u}{\partial \delta}\right)dy + u\,(\delta)\right]\frac{d\delta}{dx} = \frac{C_p\,\nu}{N_{Pr}\,\lambda}\left(\frac{\partial T}{\partial y}\right)_o \tag{9}$$

The integrand in (9) can be obtained from (6) under the assumption of interchangability of the differential with respect to y and δ

$$\frac{\partial V_1}{\partial y} = \frac{(g/\nu)\,\{[1 + (n^2/\nu)u_1y[1 - \exp(-n^2u_1y/\nu)]\} - (n^2/\nu)(\delta - y)y\,[1 + ((n^2/\nu)\,u_1y - 1)\exp(-n^2u_1y/\nu)]\,V_1\}}{\{1 + (n^2/\nu)u_1y\,[1 - \exp(n^2u_1y/\nu)]\}^2} \tag{10}$$

where $V_1 = \partial u_1/\partial \delta$.

Case II: $\delta \leqq y^*$

One must consider two regions adjoining at y^*. The equation of motion for $y \leqq y^*$ is identical to (6); however a governing equation must be obtained from (1) and (5) for $y^* \leqq y$

$$\text{Equation (6)} \quad 0 \leqq y \leqq y^* \tag{11}$$

$$\frac{\partial^2 u_2}{\partial y^2} = \frac{-(K/\sqrt{\nu})(\partial u_2/\partial y)^2}{\sqrt{(g/\nu)(\delta - y) - \partial u_2/\partial y}}, \quad y^* \leqq y \leqq \delta \tag{12}$$

where u_2 denotes u in von Karman's region ($\partial u_2/\partial y \geqq 0$ and $\partial^2 u_2/\partial y^2 \leqq 0$). The temperature profile can also be found by integrating (2) with (4) and (5)

$$\frac{T - T_w}{\Delta T} = \frac{1}{I_2} \int_o^y \frac{dy}{1 + N_{Pr}(\nu_e/\nu)\,\alpha} \tag{13}$$

where I_2 denotes the integral of (13) from 0 to δ, in which ν_e takes either (4) or (5).

The corresponding integrand of (9) will be derived from (11) and (12) to give

$$\text{Equation (10)} \quad 0 \leqq y \leqq y^* \tag{14}$$

$$\frac{\partial^2 V_2}{\partial y^2} = \frac{-\frac{K}{\sqrt{\nu}}\left\{ 2\left[\frac{g}{\nu}(\delta - y) - \frac{\partial u_2}{\partial y}\right]\left(\frac{\partial u_2}{\partial y}\right)\left(\frac{\partial V_2}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial u_2}{\partial y}\right)^2\left[\frac{g}{\nu} - \frac{\partial V_2}{\partial y}\right]\right\}}{\left[\frac{g}{\nu}(\delta - y) - \frac{\partial u_2}{\partial y}\right]^{3/2}} \qquad y^* \leqq y \leqq \delta \tag{15}$$

where $V_2 = \partial u_2/\partial \delta$.

METHOD OF SOLUTION

Since all the equations developed in the previous section are too cumbersome to be treated analytically, the author shall take recourse to the numerical method. By substitution of either (8) or (13) Equation (9) becomes

$$\left[\int_o^\delta (\partial u/\partial \delta)dy + u(\delta)\right] I_{1,2}\frac{d\delta}{dx} = \frac{(C_p\,\Delta T/\lambda)\nu}{N_{Pr}} \tag{16}$$

where subscripts 1 and 2 correspond to cases I and II, respectively. In essence the mathematical problem is to find a solution of first-order differential equation $d\delta/dx = F(\delta)$ for $\delta = 0$ at $x = 0$. A subtle complication arises because u must be known to evaluate $F(\delta)$.

From the numerical standpoint the following scheme was adopted for convenience.

Integrate (16) to obtain

$$\int_o^\delta \left[\int_o^\delta (\partial u/\partial \delta)dy + u(\delta)\right] I_{1,2}\; d\delta = \frac{(C_p\,\Delta T/\lambda)\nu\, x}{N_{Pr}} \tag{17}$$

For a given δ the corresponding x can be obtained, or vice versa. This type of numerical method seems to be amenable from the separable character of the differential equation.

The author now briefly returns to the discussion of evaluating the integrand of (17). No further discussion is necessary for Case I owing to the regular behavior of (6) and (10). On the other hand for Case II u must be obtained from (11) and (12) and $V_{1,2}$ from (14) and (15). Equation (12) can be expressed in a corresponding system of first-order differential equations:

$$\frac{\partial u_2}{\partial y} = P \tag{18}$$

$$;y^* \leqq y \leqq \delta$$

$$\frac{\partial P}{\partial y} = \frac{-(K/\sqrt{\nu})\,P^2}{\sqrt{(g/\nu)(\delta - y) - P}} \tag{19}$$

Usually a system of two first-order differential equations can satisfy only two initial (or boundary) conditions. However it has been shown (5) that the following three necessary conditions can be fulfilled because (19) has a nodal singular point at $y = \delta$:

$$u_2 = u_1 \qquad y = y^*$$
$$P = \partial u_1/\partial y \qquad y = y^*$$
$$P = 0 \qquad y = \delta$$

The exact solution has a continuous first derivative for all y and has a vanishing gradient at $y = \delta$. Therefore the traditional physical inconsistencies on velocity profile are now removed in toto.

Similarly by letting $\partial V_2/\partial y = Q$ Equation (15) can also be expressed as a system for which the following conditions can be satisfied:

$$V_2 = V_1 \qquad y = y^*$$
$$Q = \partial V_1/\partial y \qquad y = y^*$$
$$Q = g/\nu \qquad y = \delta$$

When one obtains δ and u, the computation of T follows from (8) or (13).

HEAT TRANSFER CORRELATION

The heat transfer coefficient for condensation can be defined as

$$h = \frac{k}{\Delta T}\left(\frac{\partial T}{\partial y}\right)_o = \frac{k}{I_{1,2}} \tag{20}$$

which is obtained from the heat balance $h\Delta T = k(\partial T/\partial y)_o$. In order to avoid the dependency on the plate length due to the absence of similarity (see reference *1, 4, 7*), introduce the following parameters, Reynolds number

$$N_{Re} = (4/\nu)\int_o^\delta u\,dy \,\Big|_{\text{at } x = L}$$

and average heat transfer coefficient

$$(\nu^2/g)^{1/3}h_{av}/k = (\nu^2/g)^{1/3}\,(1/L)\int_o^L dx/I_{1,2}$$

For Nusselt's laminar problem N_{Re} and h_{av} are related as

$$(\nu^2/g)^{1/3}\,h_{av}/k = 1.47\,N_{Re}^{-1/3} \tag{21}$$

RESULTS AND DISCUSSION

For wide ranges of N_{Pr} and $C_p\Delta T/\lambda$ the computation of $u(x, y)$, $T(x, y)$, and $\delta(x)$ was performed numerically

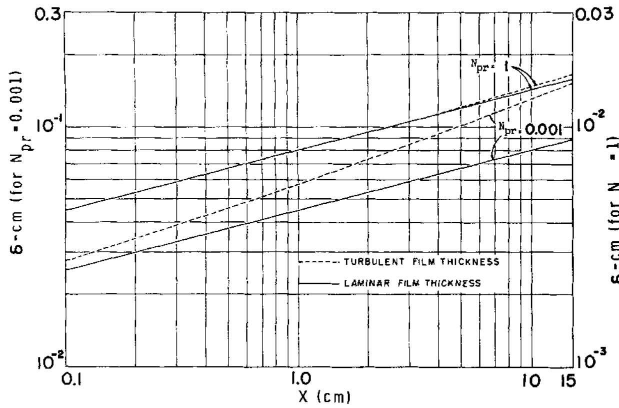

Fig. 2. Film thickness ($C_p\Delta T/\lambda = 0.04$).

with $L = 15$ cm. and $\nu = 0.005$ sq.cm./sec.* From the previous work (5) $n = 0.124$, $K = 0.4$, and $y^* = 23\nu/\sqrt{g\delta}$ were adopted so that the logarithmic velocity distribution is preserved for a positive, huge interfacial shear stress.

The turbulent film thickness $\delta(x)$ was plotted in Figure 2 for two different values of N_{Pr} and comparison was made with the corresponding laminar $\delta_n(x)$ $(= [4(C_p\Delta T/\lambda)\ x/N_{Pr}(g/\nu^2)]^{1/4})$. For $N_{Pr} = 1.0$, $\delta(x)$ deviates from $\delta_n(x)$ insignificantly because $\delta < y^*$. But for $N_{Pr} = 0.001$ the condition of $\delta > y^*$ prevails so that $\delta(x)$ is much larger than $\delta_n(x)$ owing to turbulent mixing. In general the addition of turbulent transports contributes to the increase of $\delta(x)$ in all cases. Qualitative information can be deduced by simply letting $\delta \sim C\ x^{1/4}$; then $y^* \sim C^{-1/2}x^{-1/8}$ (C is a numerical value). C is small and y^* lies above $\delta(x)$ for $N_{Pr} = 1.0$ as shown in Figure 3*a*, but for $P_r = 0.001$, y^* generally lies below $\delta(x)$ as in Figure 3*b*.

Some typical velocity and temperature profiles for $C_p\Delta T/\lambda = 0.04$ and $x = 15$ cm. are presented in Figures 4 and 5. In Figure 4 for $N_{Pr} = 10$ and 1 the milder influence of turbulent transports is manifested by nearly parabolic profiles, but as N_{Pr} decreases, the flatter velocity profiles are attributable to violent turbulent mixing. The corresponding temperature profiles follow the general trend with respect to N_{Pr} as in the previous works (*8, 12*).

For $\alpha = 1$ Figure 6 shows the correlation of average heat transfer coefficient and Reynolds number. The average heat transfer coefficients for $N_{Pr} > 0.1$ are larger than the laminar case, but a decrease is observed for smaller N_{Pr}'s. It seems quite reasonable that h_{av} will approach a limit as N_{Pr} becomes extremely small. Indeed α is an index of the ignorance in turbulent mechanism, and at present no unique information applicable to the problem seems available, other than the fact that it is dependent on the Peclet number and the flow geometry. It is generally believed that, even though not conclusive, α is greater than unity for $N_{Pr} \geqq 1$ and becomes less than unity as N_{Pr} becomes small. At any rate the reason for adopting $\alpha = 1$ was that it gives a conservative result which can be modified for other values of α. Furthermore,

* The particular choice of L and ν is immaterial to this mode of heat transfer correlation. In particular the appearance of ν could have been eliminated completely by a proper transformation.

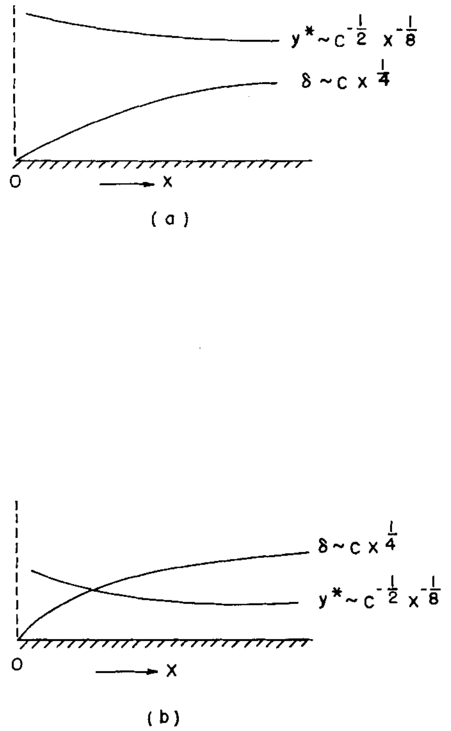

Fig. 3. Separation of condensate films.

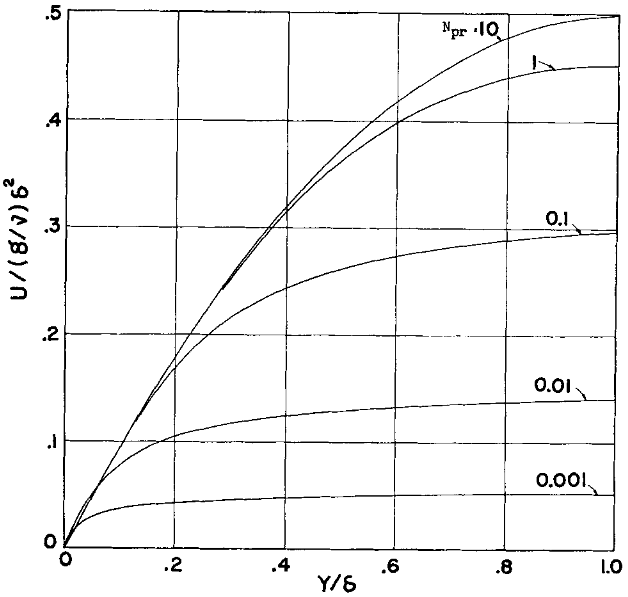

Fig. 4. Velocity ($C_p\Delta T/\lambda = 0.04$ and $X = 15$ cm.).

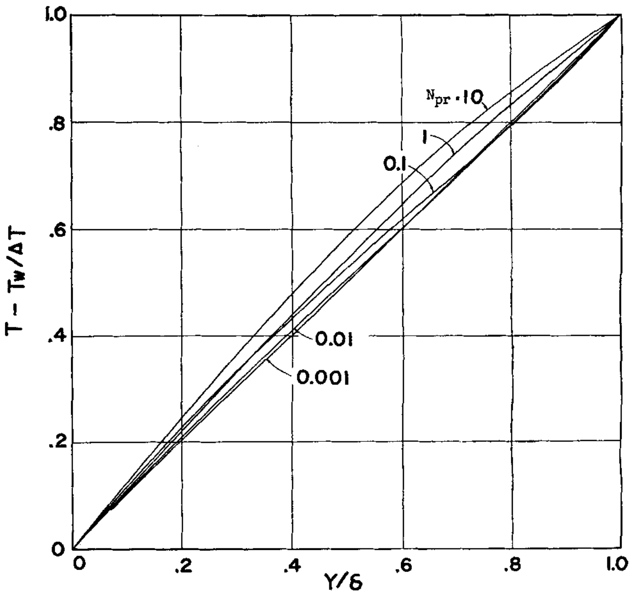

Fig. 5. Temperature ($C_p\Delta T/\lambda = 0.04$ and $X = 15$ cm.).

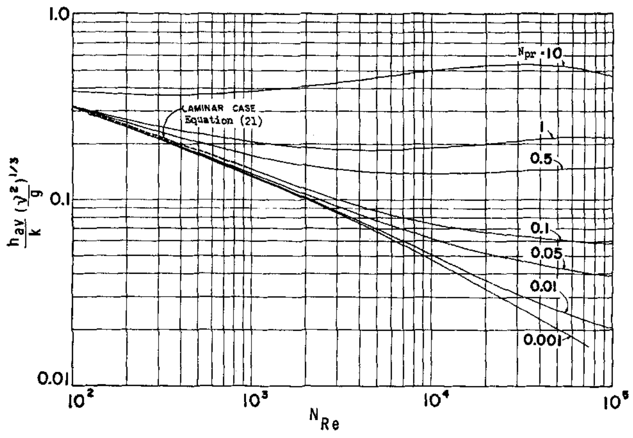

Fig. 6. Average heat transfer coefficient vs. Reynolds number.

at least for the small N_{Pr} range, the heat transfer result would not be too sensitive to α. By taking $\alpha = 0.1$ as an example h_{av} is reduced by 4.0% at $N_{Re} = 10^4$ for $N_{Pr} = 0.01$.

Figure 6 generally agrees with the qualitative result of Seban (*12*), but does not agree with Dukler's no interfacial shear stress case completely. There are legitimate reasons for expecting a rather close agreement because of the same physical model and eddy expressions involved in both analyses. At most however there can be a minor degree of discrepancy arising from the difference in computing the velocity profile, namely that of exact and approximate methods (see reference *6*).

In order to make a definite comparison between the results of Dukler and present analysis the author has carried out computations using Dukler's value of the parameters $n = 0.1$, $K = 0.4$, and $y^* = 20\nu/\sqrt{g\delta}$. Figure 7 shows Dukler's results for $N_{Pr} = 0.05$ (Figure 15 of reference *3*) and $N_{Pr} = 0.1$, 1.0, and 10.0 (Figure A-11 in Appendix of reference *3*). In the same figure the results of the author's analysis are also included for comparison. It must be noted that the decrease of Dukler's h_{av} is exceedingly drastic, so that no limit seems to exist as N_{Pr} approaches zero. Since Dukler's $N_{Pr} = 0.05$ curve passes through the aggregate of liquid metals data (*9*), his conclusion was that the turbulent transports can lower h_{av} for liquid metals. In light of the above discussion such a conclusion can not be valid. The notable discrepancy was caused by his neglecting the molecular thermal conductivity in comparison to eddy thermal conductivity, which is unjustified for liquid metals. It is discouraging because the correct analysis is still unable to predict the experimental data; however to describe the actual state of affairs of liquid metal condensation one must consider other effects which were not included in this model. Some of the

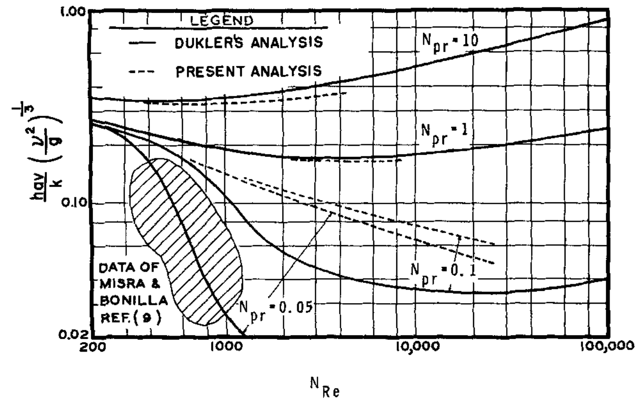

Fig. 7. Comparison of present analysis with Dukler's results.

modifications mentioned in the introduction have been investigated, and they will be presented elsewhere.

CONCLUSIONS

Turbulent condensation analysis based on the Nusselt's model indicates the following:

1. The increased heat transfer coefficients for ordinary fluids are in conformity with the experimental data; however the physical model is too naive to predict the liquid metals data.
2. The assumption adopted by Dukler regarding the magnitudes of heat transferred by the molecular conduction and turbulent mechanism is not justified; therefore his results for small Prandtl number range are physically unacceptable.

NOTATION

C_p = specific heat at constant pressure
g', g = acceleration due to gravity, $g'(\Delta\rho/\rho)$
h = condensation heat transfer coefficient
K = von Karman's constant
k = thermal conductivity
L = plate length
n = Deissler's constant
N_{Pr} = Prandtl number, (ν/κ)
N_{Re} = Reynold's number at $x = L$
$T, \Delta T$ = temperature, $T_s - T_w$
u = velocity component in x axis
x, y = coordinate system
y^* = separation of Deissler and von Karman regions

Greek Letters

α = κ_e/ν_e
δ = condensate film thickness
κ = thermal diffusivity, $(k/C_p\rho)$
λ = latent heat vaporization
ν = kinematic viscosity
$\rho, \Delta\rho$ = density, density of liquid — density of vapor
τ = shear stress

Subscripts

av = average over x
e = eddy transport coefficient
n = Nusselt's laminar case
s = saturation
w = wall

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Manuscript received June 27, 1963; revision received January 31, 1964; paper accepted February 3, 1964.